Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 6 Calculation of gradients

Find grad f. Graph some level curves f=const. Indicate ∇f by arrows at some points of these curves.

1. f = (x + 1) (2 y - 1)

In[7]:=
```
ClearAll["Global`*"]
```

In[8]:=
```
e1 = f[x_, y_] = (x + 1) (2 y - 1)
```

Out[8]=
```
(1 + x) (-1 + 2 y)
```

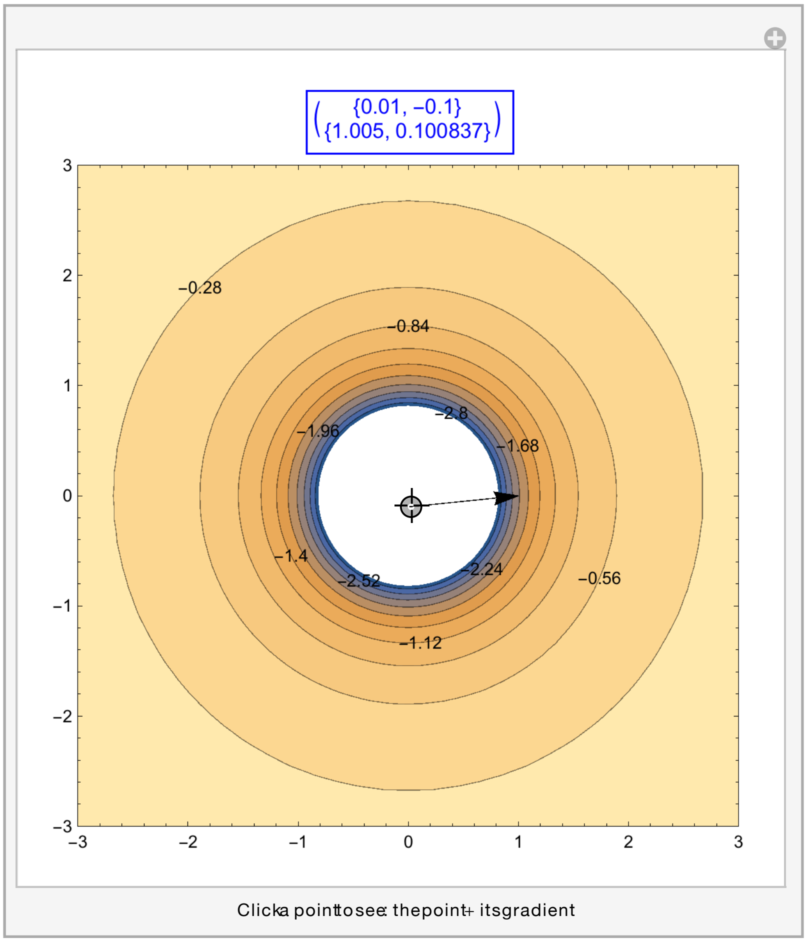
In[9]:=
```
grad[x_, y_] = Grad[f[x, y], {x, y}]
```

Out[9]=
```
{-1 + 2 y, 2 (1 + x)}
```

Below: the interactive plot was found in Mathematica documentation under ‘Grad’. It seems to cover what the problem description requires in terms of finding the gradient at various points. The only drawback is that it is not possible to get a gradient value for an exact, arbitrary point.

In[10]:=
```
Manipulate[ContourPlot[f[x, y], {x, -3, 3}, {y, -3, 3},
  Epilog → Arrow[{pt, pt + grad @@ pt}], PerformanceGoal → "Quality",
  Contours → 20, PlotRange → {{-3, 3}, {-3, 3}, {-3, 3}},
  ImageSize → Medium, ContourLabels → True,
  PlotLabel → Style[Framed[{{pt}, {grad @@ pt}}], 11, Blue]],
 {{pt, {.01, -0.1}}, Locator},
 FrameLabel → "Click a point to see: the point + its gradient",
 SaveDefinitions → True]
```

Out[10]=



3. f = y/x

In[6]:=
```
ClearAll["Global`*"]
```

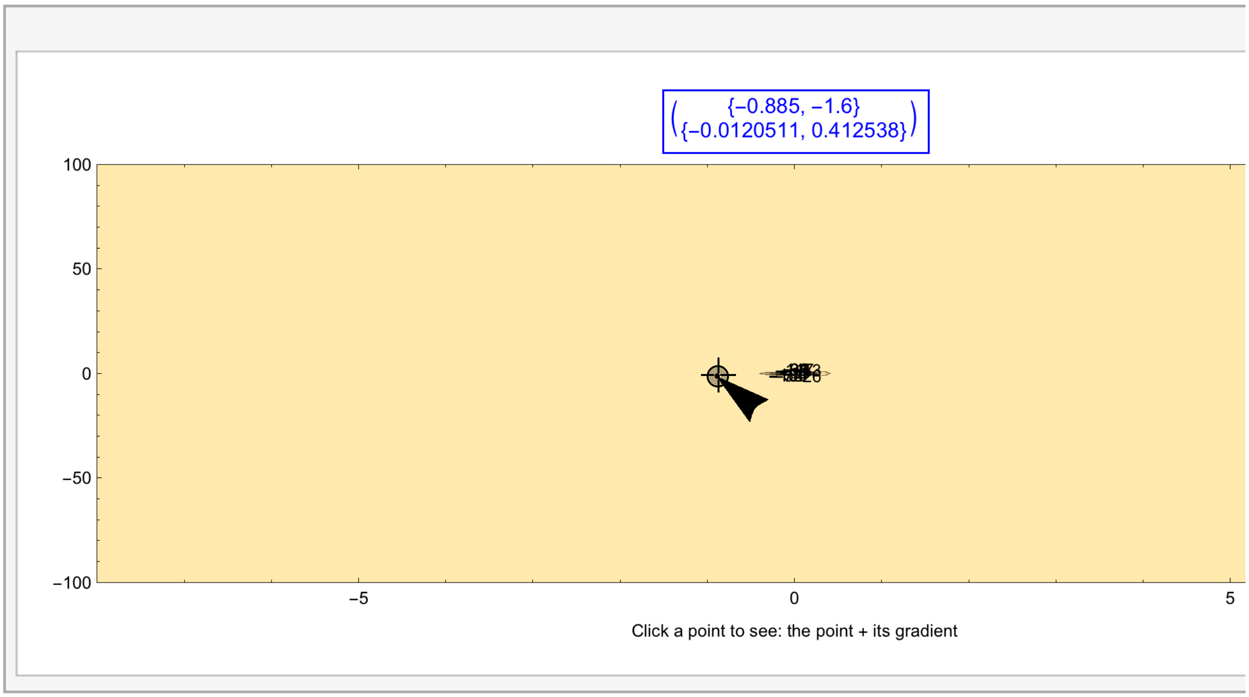
$$f[x\_, y\_] = \frac{y}{x}$$

$$\frac{y}{x}$$

```
grad[x_, y_] = Grad[f[x, y], {x, y}]
```

$\left\{-\frac{y}{x^2}, \frac{1}{x}\right\}$

```
Manipulate[ContourPlot[f[x, y], {x, -8, 8}, {y, -100, 100},
  Epilog → Arrow[{pt, pt + grad @@ pt}], PerformanceGoal → "Quality",
  Contours → 20, PlotRange → {{-8, 8}, {-100, 100}, {-140, 140}},
  ImageSize → 750, ContourLabels → True,
  PlotLabel → Style[Framed[{{pt}, {grad @@ pt}}], 11, Blue],
  FrameLabel → "Click a point to see: the point + its gradient",
  AspectRatio → .3], {{pt, {.01, -0.1}}, Locator}, SaveDefinitions → True]
```



5. $f = x^4 + y^4$

```
ClearAll["Global`*"]
```

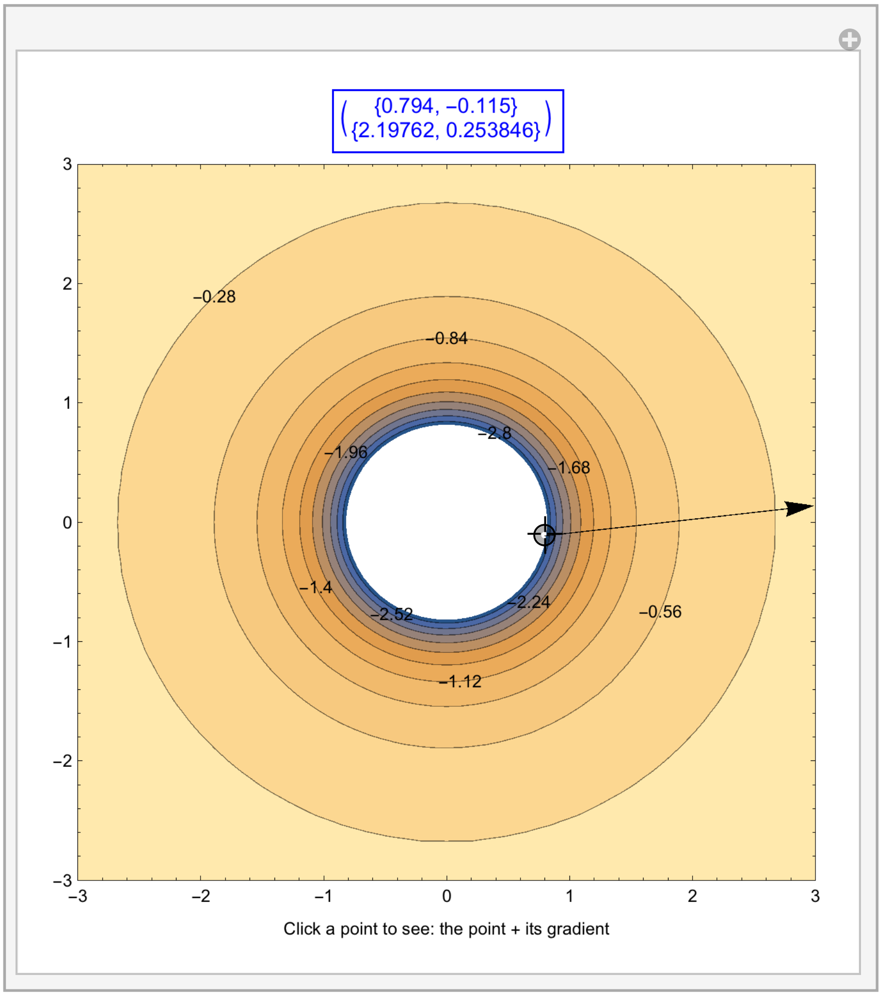
```
f[x_, y_] = x^4 + y^4
```

$x^4 + y^4$

```
grad[x_, y_] = Grad[f[x, y], {x, y}]
```

$\{4 x^3, 4 y^3\}$

```
Manipulate[ContourPlot[f[x, y], {x, -3, 3}, {y, -3, 3},
  Epilog → Arrow[{pt, pt + grad @@ pt}], PerformanceGoal → "Quality",
  Contours → 20, PlotRange → {{-3, 3}, {-3, 3}, {-3, 3}},
  ImageSize → 400, ContourLabels → True,
  PlotLabel → Style[Framed[{{pt}, {grad @@ pt}}], 11, Blue], FrameLabel →
   "Click a point to see: the point + its gradient", AspectRatio → .97],
 {{pt, {.01, -0.1}}, Locator}, SaveDefinitions → True]
```



7 - 10 Useful formulas for gradient and Laplacian
Prove and illustrate by an example.

7. $\nabla(f^n) = n\, f^{n-1}\, \nabla f$

9. $\nabla(f/g) = (1/g^2)(g\nabla f - g\nabla g)$

11 - 15 Use of gradients. Electric force.
The force in an electrostatic field given by f[x, y, z] has the direction of the gradient. Find

∇f and its value at P.

11. f = xy, P: (-4, 5)

```
ClearAll["Global`*"]
```

```
grad[x_, y_] = Grad[x y, {x, y}]
```

```
{y, x}
```

```
grad[-4, 5]
```

```
{5, -4}
```

13. $f = \text{Log}\left[x^2 + y^2\right]$, P : {8, 6}

```
ClearAll["Global`*"]
```

$\text{grad}[x\_, y\_] = \text{Grad}\left[\text{Log}\left[x^2 + y^2\right], \{x, y\}\right]$

$\left\{\frac{2\,x}{x^2 + y^2}, \frac{2\,y}{x^2 + y^2}\right\}$

```
N[grad[8, 6]]
```

```
{0.16, 0.12}
```

15. $f = 4\,x^2 + 9\,y^2 + z^2$, P : {5, -1, -1}

```
ClearAll["Global`*"]
```

$\text{grad}[x\_, y\_, z\_] = \text{Grad}\left[4\,x^2 + 9\,y^2 + z^2, \{x, y, z\}\right]$

```
{8 x, 18 y, 2 z}
```

```
grad[5, -1, -11]
```

```
{40, -18, -22}
```

18 - 23 Velocity fields

Given the velocity potential f of a flow, find the velocity **v** = ∇f of the field and its value **v**[P] at P. Sketch **v**[P] and the curve f = const passing through P.

19. f = Cos[x] Cosh[y], P : $\left(\frac{\pi}{2}, \text{Log}[2]\right)$

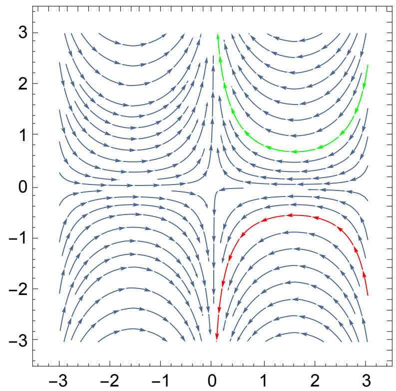
```
ClearAll["Global`*"]
```

```
velocity[x_, y_] = Grad[Cos[x] Cosh[y], {x, y}]
{-Cosh[y] Sin[x], Cos[x] Sinh[y]}
```

velocity$\left[\frac{\pi}{2}, \text{Log}[2]\right]$

$\left\{-\frac{5}{4}, 0\right\}$

StreamPlot$\left[\text{velocity}[x, y], \{x, -3, 3\}, \{y, -3, 3\}, \text{StreamPoints} \to \left\{\left\{\left\{\left\{\frac{\pi}{2}, \text{Log}[2]\right\}, \text{Green}\right\}, \{\{0.5, -1\}, \text{Red}\}, \text{Automatic}\right\}\right\}, \text{ImageSize} \to 200\right]$



21. f = e^x Cos[y], P : $\left(1, \frac{1}{2}\pi\right)$

```
ClearAll["Global`*"]
```

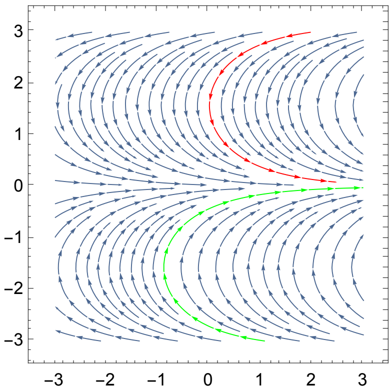
velocity[x_, y_] = Grad[e^x Cos[y], {x, y}]

{e^x Cos[y], $-e^x$ Sin[y]}

velocity$\left[1, \frac{\pi}{2}\right]$

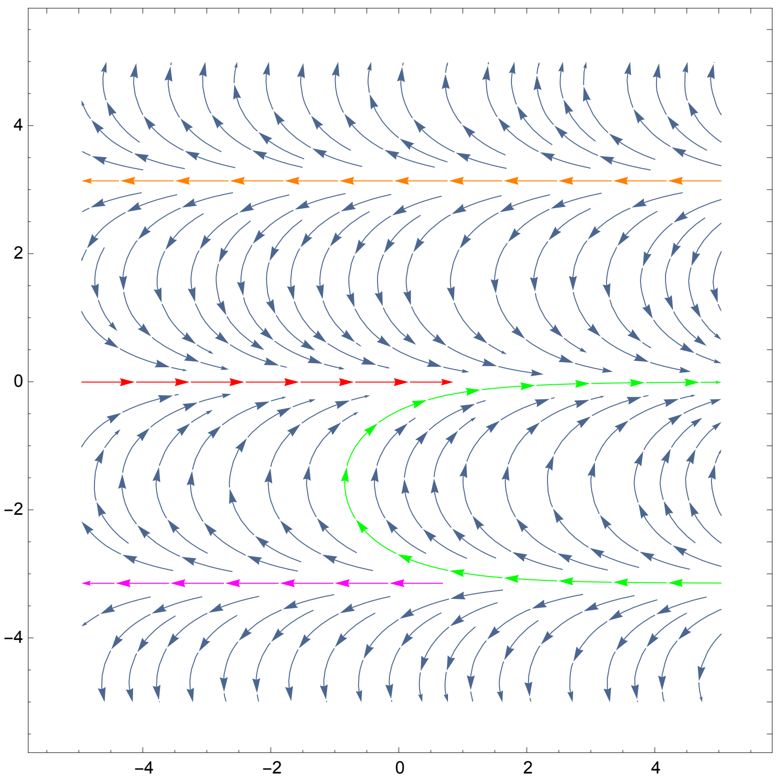
{0, $-e$}

```
StreamPlot[velocity[x, y], {x, -3, 3}, {y, -3, 3}, StreamPoints →
  {{{{0, -e}, Green}, {{0, π/2}, Red}, Automatic}}, ImageSize → 200]
```



23. At what points is the flow in problem 21 horizontal?

```
StreamPlot[velocity[x, y], {x, -5, 5}, {y, -5, 5},
 StreamPoints → {{{{0, -e}, Green}, {{0, 0}, Red}, {{0, π}, Orange},
    {{0, -π}, Magenta}, Automatic}}, ImageSize → 400]
```



Above: The only point I found that was definitely horizontal when plotted was (0, 0). The text answer is (0, $\pm n\,\pi$), so it is a more general formula. A couple of these identified designated points are shown above.

24 - 27 Heat flow.

Experiments show that in a temperature field, heat flows in the direction of maximum decrease of temperature T. Find this direction in general and at the given point P. Sketch that direction at P as an arrow.

25. $T = \frac{z}{(x^2 + y^2)}$, P : (0, 1, 2)

```
ClearAll["Global`*"]
```

teef[x_, y_, z_] = $\frac{z}{x^2 + y^2}$

$\frac{z}{x^2 + y^2}$

```
teef[0, 1, 2]
```

2

gradT[x_, y_, z_] = Grad[$-\frac{z}{(x^2 + y^2)}$, {x, y, z}]

$\left\{\frac{2 x z}{(x^2 + y^2)^2}, \frac{2 y z}{(x^2 + y^2)^2}, -\frac{1}{x^2 + y^2}\right\}$

```
gradT[0, 1, 2]
```

{0, 4, -1}

```
gradT[0, 3, 0]
```

$\left\{0, 0, -\frac{1}{9}\right\}$

```
{0, 3, 2} - %
```

$\left\{0, \frac{77}{27}, \frac{19}{9}\right\}$

The minus sign was put into the Grad expression above because I want not the maximum increase direction but the maximum decrease direction. The blue cell shows that it works with the problem point, yielding the text answer. The s.m. approached the problem of plotting by considering the isotherms at $z = 2$, the z-plane of the problem point. I copied this approach.

```
gradX[x_, y_] = {x, y, 2}
```

{x, y, 2}

Above: the function gradX is designed to insert the $z = 2$ coordinate into any point **pt** in the interactive plot.

```
gradX[0, 1]
{0, 1, 2}
```

$$\text{gradb}[x\_, y\_] = \text{Grad}\left[-\frac{2}{(x^2 + y^2)}, \{x, y\}\right]$$

$$\left\{\frac{4 x}{(x^2 + y^2)^2}, \frac{4 y}{(x^2 + y^2)^2}\right\}$$

Above: the function gradb is designed to mimic gradT in 2 dimensions. It reproduces the direction of the gradient produced by gradT projected onto the z=2 plane parallel to the xy-plane.

```
gradb[0, 1]
{0, 4}
```

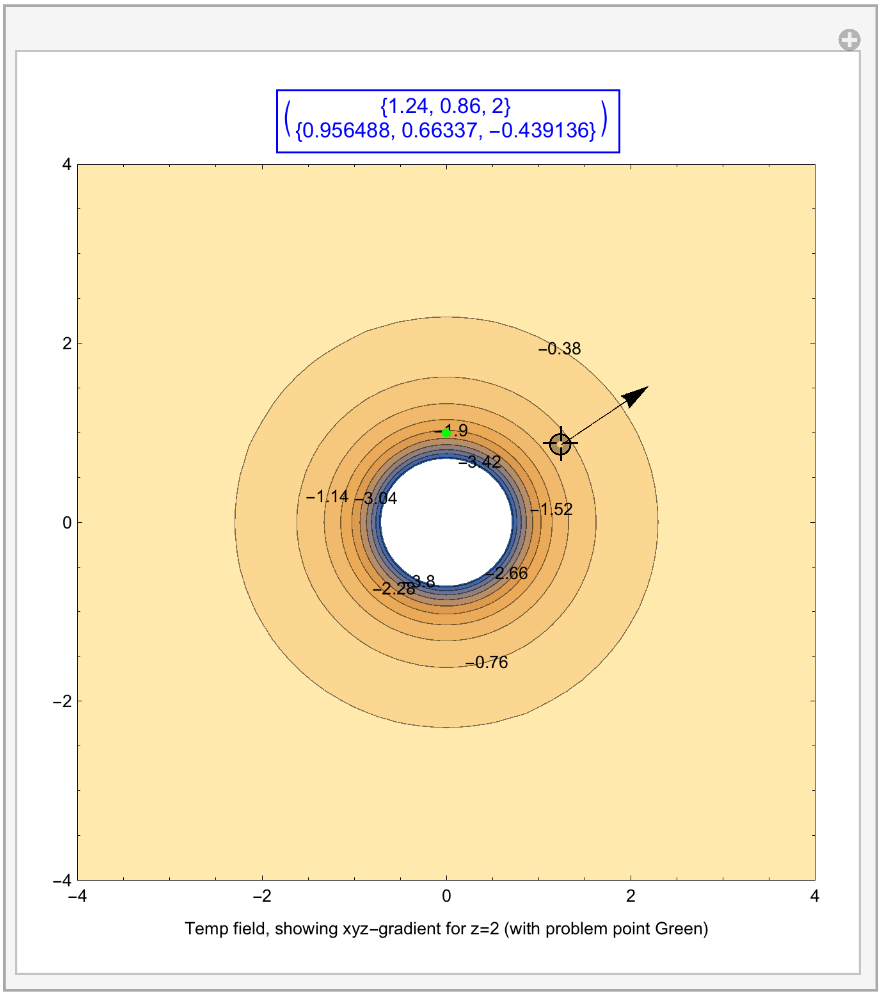
Above: testing gradb on the problem point.

$$f[x\_, y\_] = -\frac{2}{(x^2 + y^2)}$$

$$-\frac{2}{x^2 + y^2}$$

```
f[1, 1]
-1
```

```
Manipulate[ContourPlot[f[x, y], {x, -4, 4}, {y, -4, 4},
  Epilog → {Arrow[{pt, pt + gradb @@ pt}], Green, PointSize[Medium],
    Point[{0, 1}]}, PerformanceGoal → "Quality", Contours → 20,
  PlotRange → {{-4, 4}, {-4, 4}, {-4, 4}}, ImageSize → 400,
  Contours → 5, ContourLabels → True, PlotLabel →
   Style[Framed[{{gradX @@ pt}, {gradT @@ gradX @@ pt}}], 11, Blue],
  FrameLabel → "Temp field, showing xyz-gradient for z=2
     (with problem point Green)", AspectRatio → .97],
 {{pt, {.01, -0.1}}, Locator}, SaveDefinitions → True]
```




```
pointt = {1, 1, -1}
```

{1, 1, -1}

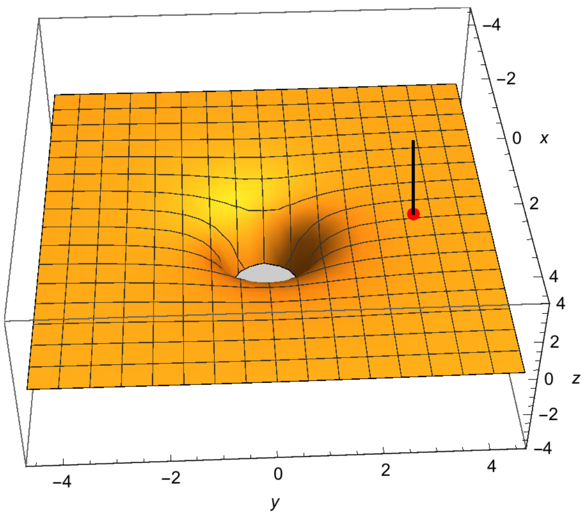
grad22[x_, y_] = Grad$\left[-\frac{2}{(x^2 + y^2)}, \{x, y\}\right]$

$\left\{\frac{4 x}{(x^2 + y^2)^2}, \frac{4 y}{(x^2 + y^2)^2}\right\}$

```
grad22[0, 3]
```

$\left\{0, \frac{4}{27}\right\}$

```
Show[{Plot3D[-2/(x^2 + y^2), {x, -4.5, 4.5}, {y, -4.5, 4.5},
   ImageSize → 300, PlotRange → {-4, 4}, AxesLabel → {x, y, z}],
  Graphics3D[{PointSize[Large], Red, Point[{0, 3, 0}],
    Black, Arrow[Tube[{{0, 3, 0}, {0, 77/27, 5}}, .03]]}]}]
```



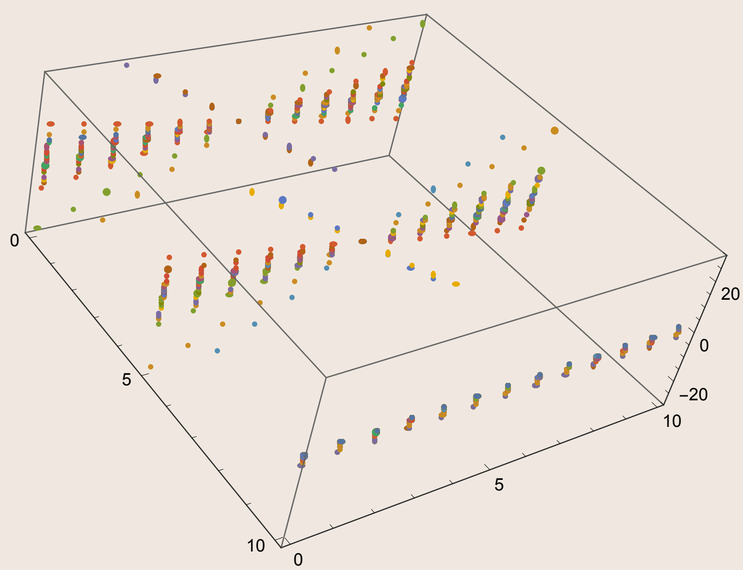

```
Arrow[Tube[{{0, 0, -4}, {0, 0, 4}}, .01]]
```

Below: some failed experiments trying to get the problem to show up in three dimensions. The ListPointPlot3D try does show how to make and use some tables as input to a plot.

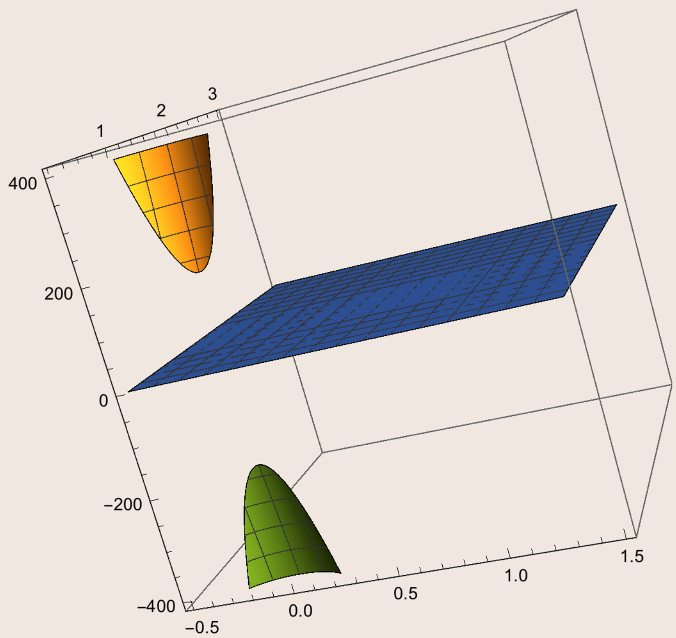
```
data1 = Flatten[Table[{2 x z/(x^2 + y^2)^2, 2 y z/(x^2 + y^2)^2, -1/(x^2 + y^2)},
    {x, -3, -0.1, 0.5}, {y, -3, 3, 0.5}, {z, -3, 3, .5}], 1];
data2 = Flatten[Table[{2 x z/(x^2 + y^2)^2, 2 y z/(x^2 + y^2)^2, -1/(x^2 + y^2)},
    {x, .1, 3, 0.5}, {y, -3, 3, 0.5}, {z, -3, 3, .5}], 1];
data3 = Flatten[Table[{2 x z/(x^2 + y^2)^2, 2 y z/(x^2 + y^2)^2, -1/(x^2 + y^2)},
    {x, -3, 3, 0.5}, {y, -3, -.1, 0.5}, {z, -3, 3, .5}], 1];
data4 = Flatten[Table[{2 x z/(x^2 + y^2)^2, 2 y z/(x^2 + y^2)^2, -1/(x^2 + y^2)},
    {x, -3, 3, 0.5}, {y, .1, 3, 0.5}, {z, -3, 3, .5}], 1];
```

```
dataall = Union[data1, data2, data3, data4];
```

```
ListPointPlot3D[dataall,
 DataRange -> {{0, 10}, {0, 10}}, PlotRange -> {-30, 30}]
```



```
ContourPlot3D[- z/(x^2 + y^2), {x, -.5, 1.5}, {y, .2, 3}, {z, -400, 400}]
```

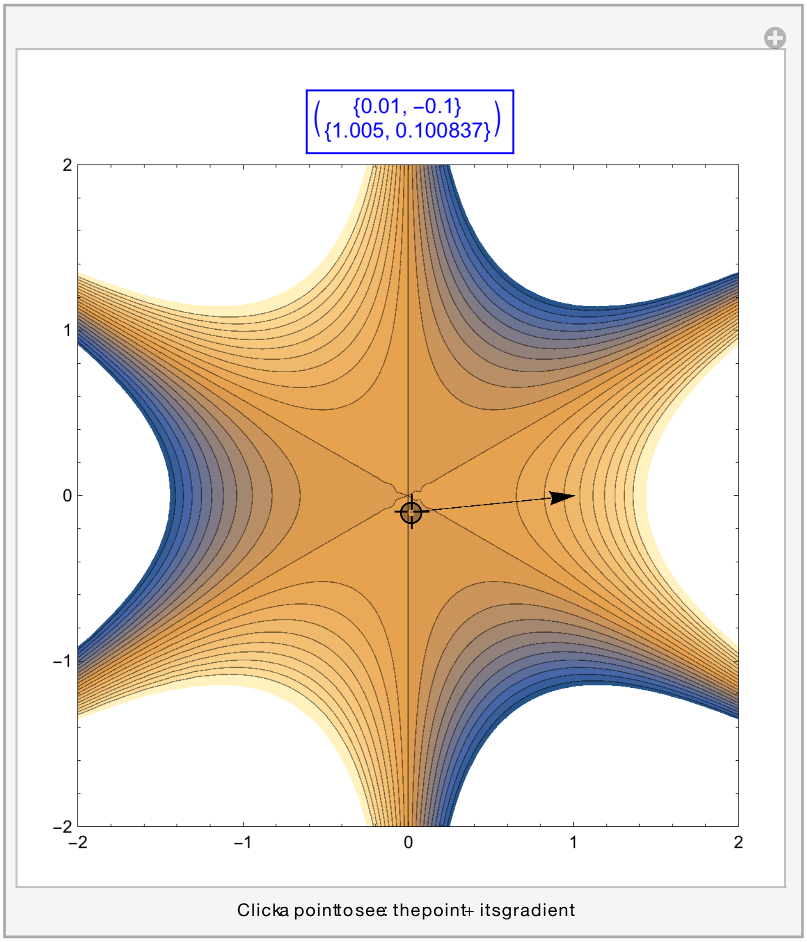



```
ContourPlot3D[- z/(x^2 + y^2), {x, -.5, 1.5}, {y, .2, 3}, {z, -400, 400}]
```

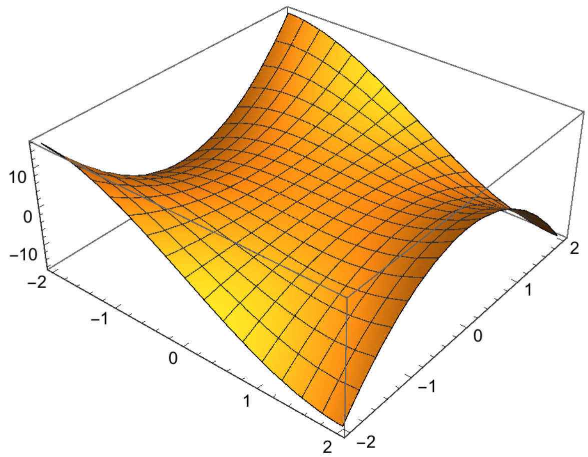

27. CAS project. Isotherms. Graph some curves of constant temperature ("isotherms") and indicate directions of heat flow by arrows when the temperature equals (a) x^3 - 3xy^2, (b) Sin[x] Sinh[y], and (c) e^x Cos[y].

```
ClearAll["Global`*"]
grad[x_, y_] = Grad[x^3 - 3 x y^2, {x, y}]
{3 x^2 - 3 y^2, -6 x y}
```

```
Manipulate[ContourPlot[x^3 - 3 x y^2, {x, -2, 2}, {y, -2, 2},
  Epilog → Arrow[{pt, pt + grad @@ pt}], PerformanceGoal → "Quality",
  Contours → 20, PlotRange → {{-2, 2}, {-2, 2}, {-3, 3}},
  ImageSize → Medium, ContourLabels → Automatic,
  PlotLabel → Style[Framed[{{pt}, {grad @@ pt}}], 11, Blue]],
 {{pt, {.01, -0.1}}, Locator},
 FrameLabel → "Click a point to see: the point + its gradient",
 SaveDefinitions → True]
```



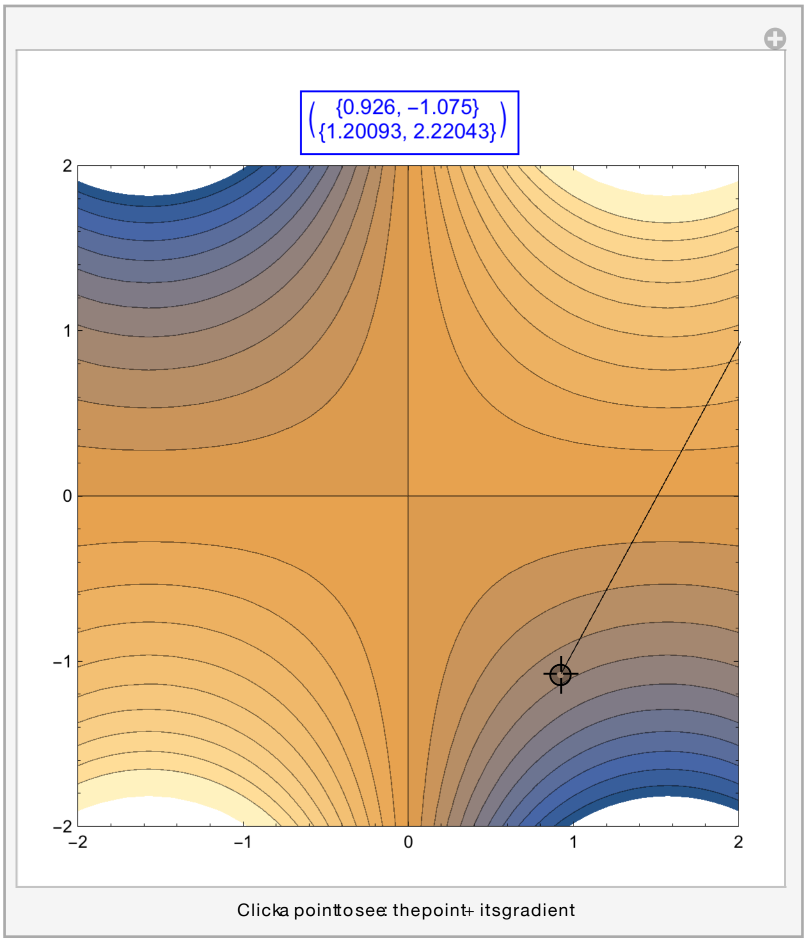
```
Plot3D[x^3 - 3 x y^2, {x, -2, 2}, {y, -2, 2}, ImageSize → 300]
```



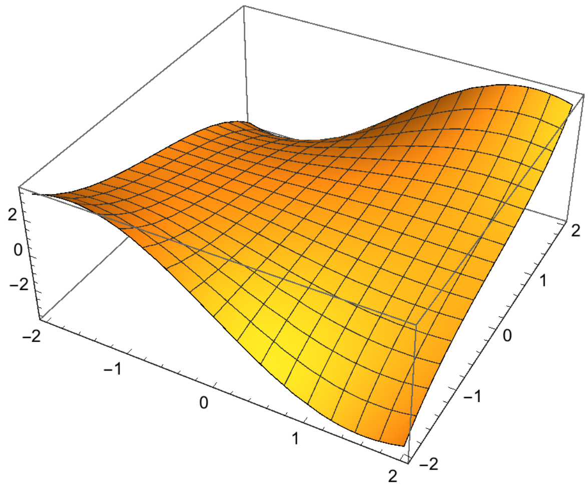

```
ClearAll["Global`*"]
```

```
grad[x_, y_] = Grad[Sin[x] Sinh[y], {x, y}]
{Cos[x] Sinh[y], Cosh[y] Sin[x]}
```

```
Manipulate[ContourPlot[Sin[x] Sinh[y], {x, -2, 2}, {y, -2, 2},
  Epilog → Arrow[{pt, pt + grad @@ pt}], PerformanceGoal → "Quality",
  Contours → 20, PlotRange → {{-2, 2}, {-2, 2}, {-3, 3}},
  ImageSize → Medium, ContourLabels → Automatic,
  PlotLabel → Style[Framed[{{pt}, {grad @@ pt}}], 11, Blue]],
 {{pt, {.01, -0.1}}, Locator},
 FrameLabel → "Click a point to see: the point + its gradient",
 SaveDefinitions → True]
```



```
Plot3D[Sin[x] Sinh[y], {x, -2, 2}, {y, -2, 2}, ImageSize → 300]
```

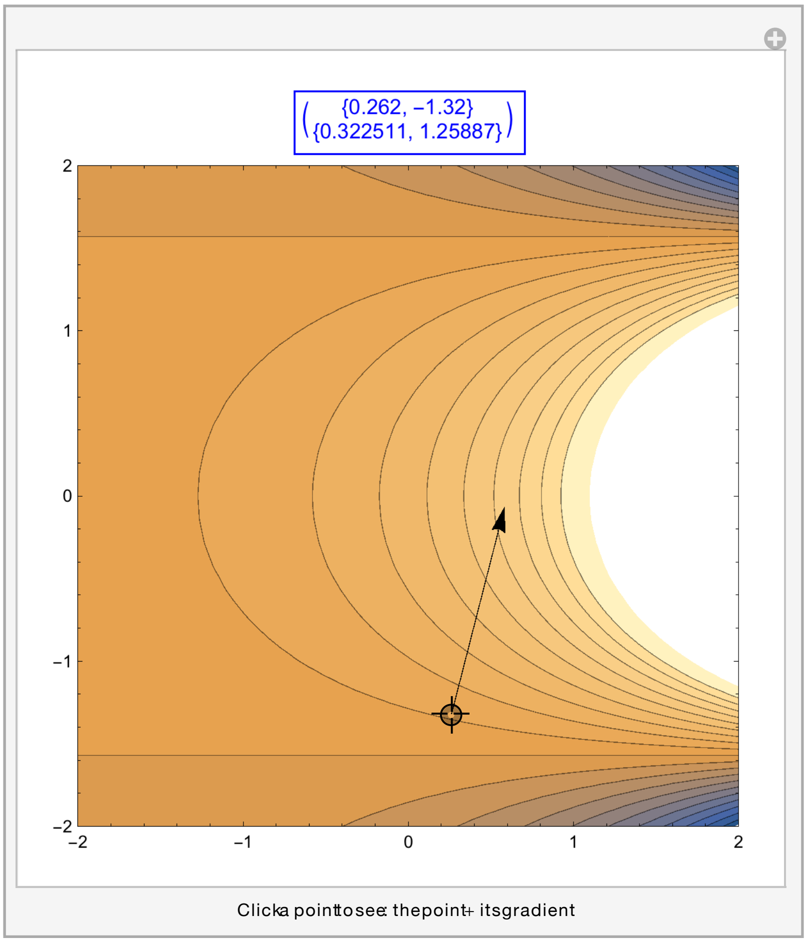



```
ClearAll["Global`*"]
```

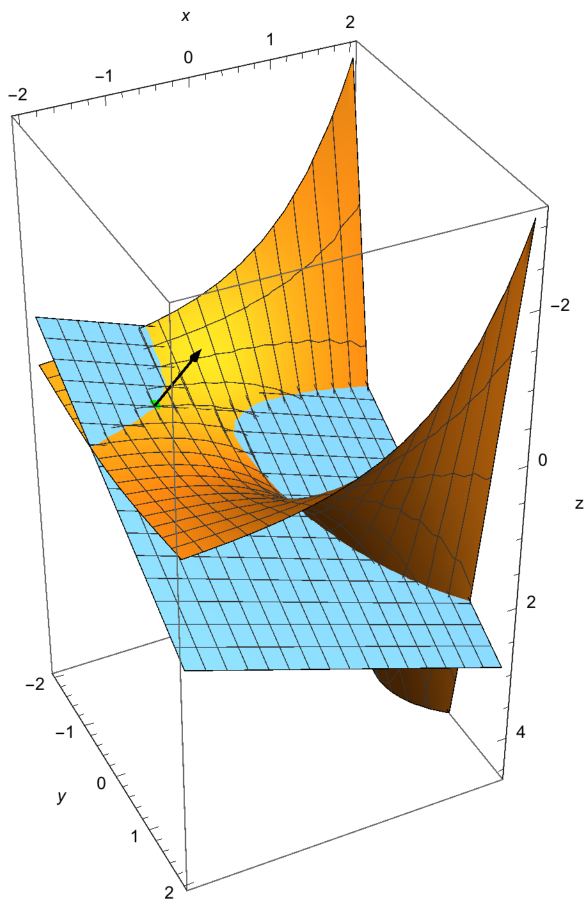
```
grad[x_, y_] = Grad[ⅇ^x Cos[y], {x, y}]
```

$\{e^x \text{Cos}[y], -e^x \text{Sin}[y]\}$

```
Manipulate[ContourPlot[ⅇ^x Cos[y], {x, -2, 2}, {y, -2, 2},
  Epilog → Arrow[{pt, pt + grad @@ pt}], PerformanceGoal → "Quality",
  Contours → 20, PlotRange → {{-2, 2}, {-2, 2}, {-3, 3}},
  ImageSize → Medium, ContourLabels → Automatic,
  PlotLabel → Style[Framed[{{pt}, {grad @@ pt}}], 11, Blue]],
 {{pt, {.01, -0.1}}, Locator},
 FrameLabel → "Click a point to see: the point + its gradient",
 SaveDefinitions → True]
```



```
Show[Plot3D[{{ⅇ^x Cos[y]}, {(1 + x)/ⅇ + Cos[1]/ⅇ + (1 + y) Cos[1]}},
  {x, -2, 2}, {y, -2, 2}, ImageSize → 300,
  AxesLabel → {x, y, "z"}, BoxRatios → Automatic],
 Graphics3D[{PointSize[Large], Green, Point[{-1, -1, ⅇ^-1 Cos[-1]}],
   Black, Arrowheads[{{.02, 1}}], Arrow[Tube[{{-1, -1, ⅇ^-1 Cos[-1]},
      {1/ⅇ - 1, Cos[1] - 1, -1 + ⅇ^-1 Cos[-1]}}, .015]]}]]
```



According to *MathWorld*, the equation for a tangent plane is:
$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. At the point (-1,-1),

```
z = ⅇ^-1 Cos[-1] + ⅇ^-1 (x + 1) + Cos[-1] (y + 1)
```

$$\frac{1 + x}{e} + \frac{\text{Cos}[1]}{e} + (1 + y)\,\text{Cos}[1]$$

```
gradz = Grad[z, {x, y}]
```

$$\left\{\frac{1}{e}, \text{Cos}[1]\right\}$$

As the above shows, this works.

Again, from *MathWorld*, the equation for a normal vector at a point x_0, y_0 on *a* surface $z = f(x, y)$ is given by

$$N = \left| \begin{matrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{matrix} \right|,$$

(the vector) where f_x and f_y are partial derivatives.

```
nN = {D[(1 + x)/ⅇ + Cos[1]/ⅇ + (1 + y) Cos[1], {x}],
  D[(1 + x)/ⅇ + Cos[1]/ⅇ + (1 + y) Cos[1], {y}], -1}
{1/ⅇ, Cos[1], -1}
```

Adding this to the arrow starting point works now, thanks to a Stack Exchange tip on the use of BoxRatio to get the axes equalized.

****

All the problems after no. 29 were omitted from the PDF version of the text.

33. Find the normal to the ellipsoid surface $6x^2 + 2y^2 + z^5 = 225$, first in general expression, then at the point $P = (5, 5, 5)$. Find the unit normal.

```
ClearAll["Global`*"]
e1 = 6 x^2 + 2 y^2 + z^2
6 x^2 + 2 y^2 + z^2
```

Above: It looks like the constant should be dropped, it just gums up the works.

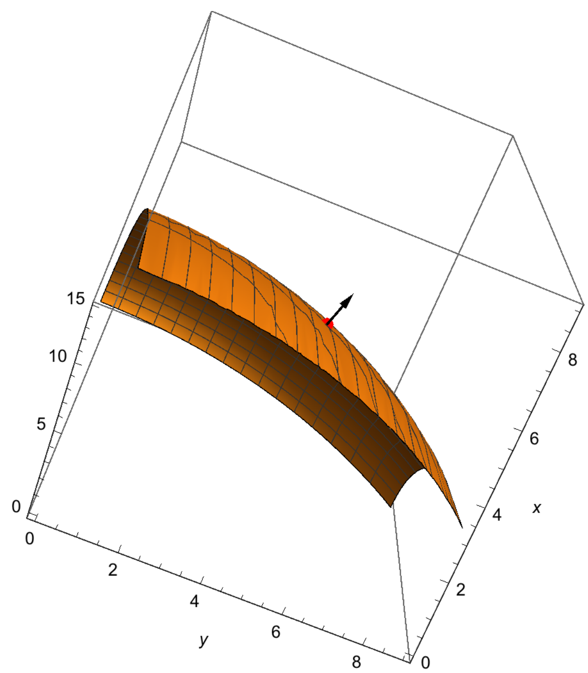
```
e2[x_, y_, z_] = Grad[e1, {x, y, z}]
{12 x, 4 y, 2 z}
e3 = e2[5, 5, 5]
{60, 20, 10}
e44 = e2[0, 0, 15]
{0, 0, 30}
e4 = Norm[e3]
10 √41
e45 = Norm[e44]
30
```

$$e5 = \frac{e3}{e4}$$

$$e51 = \frac{e44}{e45}$$

{0, 0, 1}

$$\text{Show}\left[\text{Plot3D}\left[\sqrt{225 - 6x^2 - 2y^2} == 0, \{x, 0, 9\}, \{y, 0, 9\}, \text{AxesLabel} \to \text{Automatic}, \text{BoxRatios} \to \text{Automatic}\right], \text{Graphics3D}\left[\{\text{PointSize}[\text{Large}], \text{Red}, \text{Point}[\{5, 5, 5\}], \text{Black}, \text{Arrowheads}[\{\{.02, 1\}\}], \text{Arrow}\left[\text{Tube}\left[\left\{\{5, 5, 5\}, \left\{5 + \frac{6}{\sqrt{41}}, 5 + \frac{2}{\sqrt{41}}, 5 + \frac{1}{\sqrt{41}}\right\}\right\}, .03\right]\right]\}\right]\right]$$



$$e7[x\_, y\_, z\_] = \text{Grad}\left[\sqrt{225 - 6x^2 - 2y^2} - z, \{x, y, z\}\right]$$

$$\left\{-\frac{6x}{\sqrt{225 - 6x^2 - 2y^2}}, -\frac{2y}{\sqrt{225 - 6x^2 - 2y^2}}, -1\right\}$$

e7[5, 5, 5]

{-6, -2, -1}

$$aa = \sqrt{\frac{6}{225}} \; // \; N$$

0.163299

$$bb = \sqrt{\frac{2}{225}} \text{ // N}$$

0.0942809

$$cc = \sqrt{\frac{1}{225}} \text{ // N}$$

0.0666667

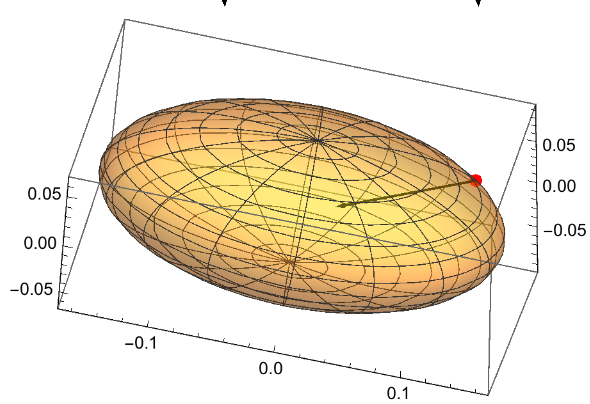
```
lgrad[u_, v_] =
 Grad[{.1633 Cos[u] Sin[v], .0943 Sin[u] Sin[v], .0666 Cos[v]}, {u, v}]
{{-0.1633 Sin[u] Sin[v], 0.1633 Cos[u] Cos[v]},
 {0.0943 Cos[u] Sin[v], 0.0943 Cos[v] Sin[u]}, {0, -0.0666 Sin[v]}}

lgrad[.2, .2]
{{-0.00644537, 0.156855}, {0.0183611, 0.0183611}, {0, -0.0132314}}
```

$$\frac{6}{\sqrt{41}}, \frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}$$

$$\sqrt{225}$$

15

```
Show[ParametricPlot3D[
  {.1633 Cos[u] Sin[v], .0943 Sin[u] Sin[v], .0666 Cos[v]}, {u, 0, 2 π},
  {v, 0, π}, PlotStyle -> Opacity[.4], BoxRatios → Automatic],
 Graphics3D[{PointSize[Large], Red, Point[
    {(.1633)^2 5, (.0943)^2 5, (.0666)^2 5}], Black, Arrowheads[{{.02, 1}}],
   Arrow[Tube[{{(.1633)^2 5, (.0943)^2 5, (.0666)^2 5},
      {(.1633)^2 6/√41, (.0943)^2 2/√41, (.0666)^2 1/√41}}, .001]]}]]
```



In the parametric form, it is hard to see the arrow angle clearly, but it might be right.